Computability, Noncomputability, and Hyperbolic Systems

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January 4, 2012

Abstract

In this paper we study the computability of the stable and unstable manifolds of a hyperbolic equilibrium point. These manifolds are the essential feature which characterizes a hyperbolic system. We show that (i) locally these manifolds can be computed, but (ii) globally they cannot (though we prove they are semi-computable). We also show that Smale's horseshoe, the first example of a hyperbolic invariant set which is neither an equilibrium point nor a periodic orbit, is computable.

1 Introduction

Dynamical systems are powerful objects which can be found in numerous applications. However, this versatility does not come without cost: dynamical systems are objects which are inherently very hard to study.

Recently, digital computers have been successfully used to analyze dynamical systems, through the use of numerical simulations. However, with the known existence of phenomena like the "butterfly effect" – a small perturbation on initial conditions can be exponentially amplified along time – and the fact that numerical simulations always involve some truncation errors, the reliability of such simulations for providing information about the long-term evolution of the systems is questionable.

For instance, numerical simulations suggested the existence of a "strange" attractor for the Lorenz system [Lor63], and it was widely believed that such an attractor existed. However, the formal proof of its existence remained elusive, being at the heart of the 14th from the list of 18 problems that the Fields medalist S. Smale presented for the new millennium [Sma98]. Finally, after a 35 years

hiatus, a computer-aided formal proof was achieved in [Tuc98], [Tuc99], but this example painstakingly illustrates the difference between numerical evidence and a full formal proof, even if computer-based.

For the above reasons, it is important to understand which properties can be accurately computed with a computer, and those which cannot. We have previously dwelled on this subject on our paper [GZar] which focused on "stable" dynamical systems having hyperbolic attractors. The latter systems were extensively studied in the XXth century and were thought to correspond to the class of "meaningful" dynamical systems. This happened for several reasons: the "stability" – structural stability – which implies robustness of behavior to small perturbations was believed to be of uttermost importance, since it was thought that only such systems could exist in nature; there were also results (Peixoto's Theorem [Pei62]) which showed that, in the plane, these systems are dense and their invariant sets (which include all attractors) can be fully characterized (they can only be points or periodic orbits).There was hope that such results would generalize for spaces of higher dimensions, but it was shattered by S. Smale, who showed that there exist "chaotic" hyperbolic invariant sets which are neither a point nor a periodic orbit (Smale's horseshoe [Sma67]) and that for dimensions ≥ 3 structurally stable systems are not dense [Sma66].

Moreover, the notion of structural stability was shown to be too strong a requirement. In particular, the Lorenz attractor, which is embedded in a system modeling weather evolution, is not structurally stable, although it is stable to perturbations in the parameters defining the system [Via00] (in other words, robustness may not be needed for *all* mathematical properties of the system).

Despite the fact that systems with hyperbolic attractors are no longer regarded as "the" meaningful class of dynamical systems for spaces of dimension $n \geq 3$, they still play a central role in the study of dynamical systems. In essence, what characterizes a hyperbolic set is the existence at each point of an invariant splitting of the tangent space into stable and unstable directions, which generate the local stable and unstable manifolds (see Section 2).

In this paper we will study the computability of the stable and unstable manifolds for hyperbolic equilibrium points. The stable and unstable manifolds are constructs which derive from the stable manifold theorem. This theorem states that for each hyperbolic equilibrium point x_0 (see Section 2 for a definition), there exists a manifold S such that a trajectory on S will converge to x_0 at an exponential rate as $t \to +\infty$. Moreover every trajectory converging to x_0 lies entirely in S. S is call a stable manifold. Similarly one can obtain the unstable manifold U, using similar conditions when $t \to -\infty$. (The stable and unstable manifolds of a hyperbolic equilibrium point are depicted in Fig. 1.) It is important to note that classical proofs on existence of S and U are non-constructive. Thus computability (or, for the matter, semi-computability) of the stable/unstable manifold does not follow straightforwardly from these proofs.

We address the following basic question: given a dynamical system and some hyperbolic equilibrium point, can we compute its stable and unstable manifolds? We will show that the answer is twofold: (i) we can compute the stable and unstable manifolds S and U given by the stable manifold theorem which are, in some sense, local, since there are (in general) trajectories starting in points which do not lie in S that will converge to x_0 (the set of all these point united with S would yield the global stable manifold – see Section 2), but (ii)

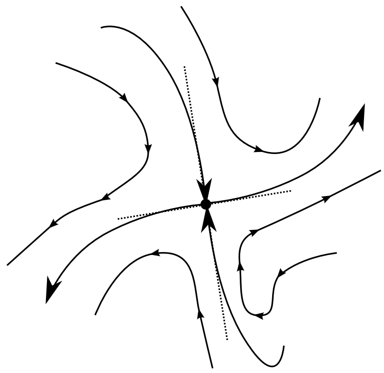

Figure 1: Dynamics near a hyperbolic equilibrium point.

the global stable and unstable manifolds are not, in general, computable from the description of the system and the hyperbolic point. Since classical proofs on existence of S and U are non-constructive, a different approach is needed if one wishes to construct an algorithm that computes S and U. Our approach aiming at a constructive proof makes use of function-theoretical treatment of the resolvents (see the first paragraph of Section 4 for more details).

A most likely interpretation for these results is that, since the definition of hyperbolicity is local, computability also applies locally. However global homoclinic tangles can occur [GH83], leading to chaotic behavior for such systems. So it should not be expected that the global behavior of stable and unstable manifolds be globally computable in general, as indeed our results show.

In the end of the paper we also show that the prototypical example of an hyperbolic invariant set which is neither a fixed point nor a periodic orbit – the Smale horseshoe – is computable.

The computability of simple attractors – hyperbolic periodic orbits and equilibrium points – was studied in our previous papers [Zho09], [GZar] for planar dynamics, as well as the computability of their respective domains of attraction.

A number of papers study dynamical systems, while not exactly in the context used here. For example the papers [Moo90], [BBKT01], [Col05], [BY06], [Hoy07] provide interesting results about the long-term behavior of dynamical systems using, among others, symbolic dynamics or a statistical approach. See [BGZar] for a more detailed review of the literature.

2 Dynamical systems

In short, a dynamical system is a pair consisting of a state space where the action occurs and a function f which defines the evolution of the system along time. See [HS74] for a precise definition. In general one can consider two kinds of dynamical systems: discrete ones, where time is discrete and one obtains the evolution of the system by iterating the map f, and continuous ones, where the evolution of the system along time is governed by a differential equation of the type

$$\dot{x} = f(x) \tag{1}$$

where t is the independent variable (the "time") and $\dot{x}$ denotes the derivative $dx(t)/dt$. Continuous-time systems can be translated to discrete-time using time-one maps and, in some cases, the Poincaré map, and vice-versa (using the suspension method). Therefore to study dynamical systems one can focus on continuous-time ones.

Although the essential feature of hyperbolic attractors is the existence of an invariant splitting of the tangent space into stable and unstable directions generating the local stable and unstable manifolds, for a hyperbolic equilibrium point x_0, it can be described equivalently in terms of the linearization of the flow around x_0. We recall that x_0 is an equilibrium point of (1) iff $f(x_0) = 0$.

Definition 1 *An equilibrium point x_0 of (1) is hyperbolic if none of the eigenvalues of $Df(x_0)$ has zero real part.*

According to the value of the real part of these eigenvalues, one can determine the behavior of the linearized flow near x_0: if the eigenvalue has negative real part, then the flow will converge to x_0 when it follows the direction given by the eigenvector associated to this eigenvalue. A similar behavior will happen when the eigenvalue has positive real part, with the difference that convergence happens when $t \to -\infty$. See Fig. 1. The space generated by the eigenvectors associated to eigenvalues of $Df(x_0)$ with negative real part is called the stable subspace $E^s_{Df(x_0)}$, and the space generated by the eigenvectors associated to eigenvalues with positive real part is called the unstable subspace $E^u_{Df(x_0)}$.

We now state the Stable Manifold Theorem (as it appears in [Per01]).

Proposition 2 (Stable Manifold Theorem) *Let E be an open subset of $\mathbb{R}^n$ containing the origin, let $f \in C^1(E)$, and let ϕ_t be the flow of the system (1). Suppose that $f(0) = 0$ and that $Df(0)$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part (*i.e. *0 is a hyperbolic equilibrium point). Then there exists a k-dimensional differentiable manifold S tangent to the stable subspace $E^s_{Df(0)}$ such that for all $t \geq 0$, $\phi_t(S) \subseteq S$ and for all $x_0 \in S$*

$$\lim_{t \to +\infty} \phi_t(x_0) = 0;$$

and there exists an $n-k$ dimensional differentiable manifold U tangent to the unstable subspace $E^u_{Df(0)}$ such that for all $t \leq 0$, $\phi_t(S) \subseteq S$ and for all $x_0 \in U$

$$\lim_{t \to -\infty} \phi_t(x_0) = 0.$$

From the local stable and unstable manifolds given by the Stable Manifold Theorem, one can define the *global stable and unstable manifolds* of (1) at a hyperbolic equilibrium point x_0 by

$$W^s_f(x_0) = \bigcup_{j=0}^{\infty} \phi_{-j}(S) \qquad (2)$$

$$W^u_f(x_0) = \bigcup_{j=0}^{\infty} \phi_j(U),$$

respectively, where

$$\phi_t(A) = \{x(t) | x \text{ is a solution of (1) with } x(0) \in A\}.$$

We note that $W^s(x_0)$ and $W^u(x_0)$ are F_σ-sets of $\mathbb{R}^n$ (recall that a subset $F \subseteq \mathbb{R}^n$ is called an F_σ-set if $F = \bigcup_{j=0}^{\infty} A_j$, where A_j, $j \in \mathbb{N}$, is a closed subset of $\mathbb{R}^n$).

We end this section by introducing Smale's horseshoe [Sma67]. We refer the reader to [GH83] for a more thorough discussion of this set. In essence, Smale's horseshoe appears when we consider a map f defined over $S = [0,1]^2$. This map is bijective and performs a linear vertical expansion of S, and a linear horizontal contraction of S, by factors $\mu > 1$ and $0 < \lambda < 1$, respectively, followed by a folding.

Definition 3 *In the conditions defined above (see [GH83] or [HSD04] for precise definitions), the Smale horseshoe is the set* Λ *given by*

$$\Lambda = \bigcap_{j=-\infty}^{+\infty} f^j(S).$$

This set is invariant for the function f *(i.e.* $f(\Lambda) = \Lambda$*).*

3 Computable analysis

Now that we have introduced the main concepts of dynamical systems theory we will use, we need to introduce concepts related to computability. The theory of computation can be rooted in the seminal work of Turing, Church, and others, which provided a framework in which to achieve computation over discrete identities or, equivalently, over the integers.

However, this definition was not enough to cover computability over continuous structures, and was then developed by other authors such as Turing himself [Tur36], Grzegorczyk [Grz57], or Lacombe [Lac55] to originate *computable analysis.*

The idea underlying computable analysis to compute over a set A is to encode each element a of A by a countable sequence of symbols from a finite alphabet (called a *ρ-name* for a). Each sequence can encode at most one element of A. The more elements we have from a sequence encoding a, the more precisely we can pinpoint a. From this point of view, it suffices to work only with names when performing a computation over A. To compute with names, we use Type-2 machines, which are similar to Turing machines, but (i) have a read-only tape, where the input (i.e. the sequence encoding it) is written; (ii) have a write-only output tape, where the head cannot move back and the sequence encoding the output is written. For more details the reader is referred to [PER89], [Ko91], [Wei00].

At any finite amount of time we can halt the computation, and we will have a partial result on the output tape. The more time we wait, the more accurate this result will be. We now introduce notions of computability over $\mathbb{R}^n$.

Definition 4 *1. A sequence* $\{r_k\}$ *of rational numbers is called a* ρ*-name of a real number* x *if there are three functions* a, b *and* c *from* $\mathbb{N}$ *to* $\mathbb{N}$ *such that for all* $k \in \mathbb{N}$, $r_k = (-1)^{a(k)} \frac{b(k)}{c(k)+1}$ *and*

$$|r_k - x| \leq \frac{1}{2^k}. \tag{3}$$

2. *A double sequence $\{r_{l,k}\}_{l,k\in\mathbb{N}}$ of rational numbers is called a ρ-name for a sequence $\{x_l\}_{l\in\mathbb{N}}$ of real numbers if there are three functions a, b, c from $\mathbb{N}^2$ to $\mathbb{N}$ such that, for all $k, l \in \mathbb{N}$, $r_{l,k} = (-1)^{a(l,k)} \frac{b(l,k)}{c(l,k)+1}$ and*
$$|r_{l,k} - x_l| \leq \frac{1}{2^k}.$$

3. *A real number x (a sequence $\{x_l\}_{l\in\mathbb{N}}$ of real numbers) is called computable if it has a computable ρ-name, i.e. the functions a, b, and c are computable or, equivalently, there is a Type-2 machine that computes a ρ-name of x ($\{x_l\}_{l\in\mathbb{N}}$, respectively) without any input.*

In general, ρ-names for real numbers do not need to be exactly those described above: different definitions can provide the same set of computable points (e.g. the ρ-name could be a sequence of intervals containing x_0 with diameter $1/k$ for $k \geq 1$, etc.). A notable exception is the decimal expansion of x_0, which cannot be used since it can lead to undesirable behavior [Tur37] because this representation does not respect the topology of the space $\mathbb{R}$.

The notion of the ρ-name can be extended to points in $\mathbb{R}^n$ as follows: a sequence $\{(r_{1k}, r_{2k}, \ldots, r_{nk})\}_{k\in\mathbb{N}}$ of rational vectors is called a ρ-name of $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ if $\{r_{jk}\}_{k\in\mathbb{N}}$ is a ρ-name of x_j, $1 \leq j \leq n$. Using ρ-names, we can define computable functions.

Definition 5 *Let X and Y be two sets, where ρ-names can be defined for elements of X and Y. A function $f : X \to Y$ is computable if there is a Type-2 machine such that on any ρ-name of $x \in X$, the machine computes as output a ρ-name of $f(x) \in Y$.*

Next we present a notion of computability for open and closed subsets of $\mathbb{R}^n$ (cf. [Wei00], Definition 5.1.15). We implicitly use ρ-names. For instance, to obtain names of open subsets of $\mathbb{R}^n$, we note that the set of rational balls $B(a, r) = \{x \in \mathbb{R}^n : |x - a| < r\}$, where $a \in \mathbb{Q}^n$ and $r \in \mathbb{Q}$, is a basis for the standard topology over $\mathbb{R}^n$. Thus a sequence $\{(a_k, r_k)\}_{k\in\mathbb{N}}$ such that $E = \cup_{k=0}^{\infty} B(a_k, r_k)$ gives a ρ-name for the open set E.

Definition 6

1. *An open set $E \subseteq \mathbb{R}^n$ is called recursively enumerable (r.e. for short) open if there are computable sequences $\{a_k\}$ and $\{r_k\}$, $a_k \in \mathbb{Q}^n$ and $r_k \in \mathbb{Q}$, such that*
$$E = \cup_{k=0}^{\infty} B(a_k, r_k).$$
Without loss of generality one can also assume that for any $k \in \mathbb{N}$, the closure of $B(a_k, r_k)$, denoted as $\overline{B(a_k, r_k)}$, is contained in E.

2. *A closed subset $A \subseteq \mathbb{R}^n$ is called r.e. closed if there exists a computable sequence $\{b_k\}$, $b_k \in \mathbb{Q}^n$, such that $\{b_k\}$ is dense in A. A is called co-r.e. closed if its complement A^c is r.e. open. A is called computable (or recursive) if it is both r.e. and co-r.e.*

3. *An open set $E \subseteq \mathbb{R}^n$ is called computable (or recursive) if E is r.e. open and its complement E^c is r.e. closed.*

4. *A compact set $K \subseteq \mathbb{R}^n$ is called computable if it is computable as a closed set and, in addition, there is a rational number b such that $||x|| \leq b$ for all $x \in K$.*

In the rest of the paper, we will work exclusively with C^1 functions $f : E \to \mathbb{R}^n$, where E is an open subset of $\mathbb{R}^n$. Thus it is desirable to present an explicit ρ-name for such functions.

Definition 7 *Let $E = \bigcup_{k=0}^{\infty} B(a_k, r_k)$, $a_k \in \mathbb{Q}^n$ and $r_k \in \mathbb{Q}$, be an open subset of $\mathbb{R}^n$ (assuming that the closure of $B(a_k, r_k)$ is contained in E) and let $f : E \to \mathbb{R}^n$ be a continuously differentiable function. Then a (C^1) ρ-name of f is a sequence $\{P_l\}$ of polynomials ($P_l : \mathbb{R}^n \to \mathbb{R}^n$) with rational coefficients such that*

$$d_{C^1(E)}(f, P_l) \leq 2^{-l} \quad \textit{for all } l \in \mathbb{N}$$

where

$$d_{C^1(E)}(f, P_l) = \sum_{k=0}^{\infty} 2^{-k} \left(\frac{||f - P_l||_k}{1 + ||f - P_l||_k} + \frac{||Df - DP_l||_k}{1 + ||Df - DP_l||_k} \right)$$

and

$$||g||_k = \max_{x \in \overline{B(a_k, r_k)}} |g(x)|.$$

We observe that this ρ-name of f contains information on both f and Df in the sense that $(P_1, P_2, \ldots)$ is a ρ-name of f while $(DP_1, DP_2, \ldots)$ is a ρ-name of Df. See [ZW03] for further details.

Throughout the rest of this paper, unless otherwise mentioned, we will assume that, in (1), f is continuously differentiable on an open subset of $\mathbb{R}^n$ and we will use the above ρ-name for f.

4 Computable stable manifold theorem

The stable manifold theorem states that near a hyperbolic equilibrium point x_0, the nonlinear system

$$\dot{x} = f(x(t)) \tag{4}$$

has stable and unstable manifolds S and U tangent to the stable and unstable subspaces $\mathbb{E}_A^s$ and $\mathbb{E}_A^u$ of the linear system

$$\dot{x} = Ax \tag{5}$$

where $A = Df(x_0)$ is the gradient matrix of f at x_0. The classical proof of the theorem relies on the Jordan canonical form of A. To reduce A to its Jordan form, one needs to find a basis of generalized eigenvectors. Since the process of finding eigenvectors from corresponding eigenvalues is not continuous in general, it is a non-computable process. Thus if one wishes to construct an algorithm that computes some S and U of (4) at x_0, a different method is needed. We will make use of an analytic, rather than algebraic, approach to the eigenvalue problem that allows us to compute S and U without calling for eigenvectors. The analytic approach is based on function-theoretical treatment of the resolvents (see, *e.g.*, [SN42], [Kat49], [Kat50], and [Rob95]).

Let us first show that the stable and unstable subspaces are computable from A for the linear hyperbolic systems $\dot{x} = Ax$. We begin with several definitions. Let $\mathfrak{A}_H$ denote the set of all $n \times n$ matrices such that the linear differential

equation $\dot{x} = Ax$, $x \in \mathbb{R}^n$, is hyperbolic, where a linear differential equation $\dot{x} = Ax$ is defined to be hyperbolic if all the eigenvalues of A have nonzero real part. The Hilbert-Schmidt norm is used for $A \in \mathfrak{A}_H$: $\|A\| = \sqrt{\sum_{i=1}^n \sum_{j=1}^n |a_{ij}|^2}$, where a_{ij} is the ij^{th} entry of A. For each $A \in \mathfrak{A}_H$, define the stable subspace $\mathbb{E}_A^s$ and unstable subspace $\mathbb{E}_A^u$ to be

$$\begin{aligned}\mathbb{E}_A^s = \text{span}\,\{&v \in \mathbb{R}^n : v \text{ is a generalized eigenvector for an eigenvalue } \lambda \text{ of } A\\ &\text{with } Re(\lambda) < 0\}\end{aligned}$$

$$\begin{aligned}\mathbb{E}_A^u = \text{span}\,\{&v \in \mathbb{R}^n : v \text{ is a generalized eigenvector for an eigenvalue } \lambda \text{ of } A\\ &\text{with } Re(\lambda) > 0\}\end{aligned}$$

Then $\mathbb{R}^n = \mathbb{E}_A^s \bigoplus \mathbb{E}_A^u$. The stable subspace $\mathbb{E}_A^s$ is the set of all vectors which contract exponentially forward in time while the unstable subspace $\mathbb{E}_A^u$ is the set of all vectors which contract backward in time.

As mentioned above, the process of finding eigenvectors from corresponding eigenvalues is not computable; thus the algebraic approach to $\mathbb{E}_A^s$ and $\mathbb{E}_A^u$ is a non-computable process. Of course, this doesn't necessarily imply that it is impossible to compute $\mathbb{E}_A^s$ and $\mathbb{E}_A^u$ from A, but rather this particular classical approach fails to be computable. So even for the linear hyperbolic system $\dot{x} = Ax$, one needs a different approach to treat the stable/unstable subspace when computability concerned. The approach used below to treat the stable/unstable subspace is analytic, rather than algebraic. Let $p_A(\lambda)$ be the characteristic polynomial for A, γ_1 be any closed curve in the left half of the complex plane that surrounds (in its interior) all eigenvalues of A with negative real part and is oriented counterclockwise, and γ_2 any closed curve in the right half of the complex plane that surrounds all eigenvalues of A with positive real part, again with counterclockwise orientation. Then

$$P_A^1 \mathbb{R}^n = \mathbb{E}_A^s, \qquad P_A^2 \mathbb{R}^n = \mathbb{E}_A^u$$

where

$$P_A^1 v = \frac{1}{2\pi i}\int_{\gamma_1} (\xi I - A)^{-1} v d\xi, \quad P_A^2 v = \frac{1}{2\pi i}\int_{\gamma_2} (\xi I - A)^{-1} v d\xi$$

(See Section 4.6 of [Rob95].)

Theorem 8 *The map $H^s : \mathfrak{A}_H \to \mathcal{A}(\mathbb{R}^n) = \{X | X \subseteq \mathbb{R}^n$ is a closed subset of $\mathbb{R}^n\}$, $A \mapsto \mathbb{E}_A^s$, is computable, where $\mathfrak{A}_H$ is represented "entrywise": $\rho = (\rho_{ij})_{i,j=1}^n$ is a name of A if ρ_{ij} is a ρ-name of a_{ij}, the ijth entry of A.*

Proof. First we observe that the map $A \mapsto$ the eigenvalues of A, $A \in \mathfrak{A}_H$, is computable. Assume that $\lambda_1, \ldots, \lambda_k, \mu_{k+1}, \ldots, \mu_n$ (counting multiplicity) are eigenvalues of A with $Re(\lambda_j) < 0$ for $1 \le j \le k$ and $Re(\mu_j) > 0$ for $k+1 \le j \le n$. Then from the ρ-names of the eigenvalues, one can compute two rectangular closed curves γ_A^1 and γ_A^2, where γ_A^1 is in the left half of the complex plane that surrounds all eigenvalues λ_j with $1 \le j \le k$ and γ_A^2 is in the right half of the complex plane that surrounds all eigenvalues μ_j with $k+1 \le j \le n$, and both γ_A^1 and γ_A^2 are oriented counterclockwise. From γ_A^1 and γ_2^A one can further

computes the maps $P_A^1, P_A^2 : \mathbb{R}^n \to \mathbb{R}^n$, where $P_A^1 v = \frac{1}{2\pi i}\int_{\gamma_A^1}(\xi I - A)^{-1} v d\xi$ and $P_A^2 v = \frac{1}{2\pi i}\int_{\gamma_A^2}(\xi I - A)^{-1} v d\xi$. We note that, on the one hand, $\mathbb{E}_A^s = (P_A^2)^{-1}(\{0\})$, and on the other hand, $\mathbb{E}_A^s = \overline{\mathbb{E}_A^s} = \overline{P_A^1 \mathbb{R}^n}$, where $\overline{K}$ denotes the closure of the set K. Then by Theorem 6.2.4 of [Wei00], $\mathbb{E}_A^s$ is both r.e. and co-r.e. closed, thus computable. The same argument shows that $\mathbb{E}_A^u$ is computable from A. ■

Next we present an effective version of the stable manifold theorem. Consider the nonlinear system

$$\dot{x} = f(x(t)) \tag{6}$$

Assume that (6) defines a dynamical system, that is, the solution $x(t, x_0)$ to (6) with the initial condition $x(0) = x_0$ is defined for all $t \in \mathbb{R}$. Since the system (6) is autonomous, if p is a hyperbolic equilibrium point, without loss of generality, we may assume that p is the origin 0.

Theorem 9 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-computable function (meaning that both f and Df are computable). Assume that the origin 0 is a hyperbolic equilibrium point of (6) such that $Df(0)$ has k eigenvalues with negative real part and $n - k$ eigenvalues with positive real part (counting multiplicity), $0 < k \le n$. Let $x(t, x_0)$ denote the solution of (6) with the initial value x_0 at $t = 0$. Then there is a (Turing) algorithm that computes a k-dimensional manifold $S \subset \mathbb{R}^n$ containing 0 such that*

- *(i) For all $x_0 \in S$, $\lim_{t\to+\infty} x(t, x_0) = 0$;*
- *(ii) There are three positive rational numbers γ, ϵ, and δ such that $|x(t, x_0)| \le \gamma 2^{-\epsilon t}$ for all $t \ge 0$ whenever $x_0 \in S$ and $|x_0| \le \delta$.*

Moreover, if $k < n$, then a rational number η and a ball D with center at the origin can be computed from f such that for any solution $x(t, x_0)$ to the equation (6) with $x_0 \in D \setminus S$, $\{x(t, x_0) : t \ge 0\} \not\subset B(0, \eta)$ no matter how close the initial value x_0 is to the origin.

Proof. First we note that under the assumption that f is C^1-computable, the solution map $x : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$, $(t, a) \mapsto x(t, a)$, is computable ([GZB09]).

Let $f(x) = Ax + F(x)$, where $A = Df(0)$ and $F(x) = f(x) - Ax$. Then the equation (6) can be written in the form of

$$\dot{x} = Ax + F(x) \tag{7}$$

The first step in constructing the desired algorithm is to break the flow e^{At} governed by the linear equation $\dot{x} = Ax$ into the stable and unstable components, denoted as $I_{\Gamma_1}(t)$ and $I_{\Gamma_2}(t)$ respectively. By making use of an integral formula, we are able to show that the breaking process is computable from A. The details for the first step: Since $F(x) = f(x) - Ax$, it follows that $F(0) = 0$, $DF(0) = 0$, F and DF are both computable because f and Df are computable functions by assumption. Thus there is a computable modulus of continuity $d : \mathbb{N} \to \mathbb{N}$ such that

$$|F(x) - F(y)| \le 2^{-m}|x - y| \quad \text{whenever} \quad |x| \le 2^{-d(m)} \ \& \ |y| \le 2^{-d(m)} \tag{8}$$

Since Df is computable, all entries in the matrix A are computable; consequently, the coefficients of the characteristic polynomial $\det(A - \lambda I_n)$ of A are computable, where $\det(A - \lambda I_n)$ denotes the determinant of $A - \lambda I_n$ and I_n is the $n \times n$ unit matrix. Thus all eigenvalues of A are computable, for they are zeros of the computable polynomial $\det(A - \lambda I_n)$. Assume that $\lambda_1, \ldots, \lambda_k, \mu_{k+1}, \ldots, \mu_n$ (counting multiplicity) are eigenvalues of A with $Re(\lambda_j) < 0$ for $1 \le j \le k$ and $Re(\mu_j) > 0$ for $k+1 \le j \le n$, where $Re(z)$ denotes the real part of the complex number z. Then two rational numbers $\sigma > 0$ and $\alpha > 0$ can be computed from the eigenvalues of A such that $Re(\lambda_j) < -(\alpha + \sigma)$ for $1 \le j \le k$ and $Re(\mu_j) > \sigma$ for $k+1 \le j \le n$. We break α into two parts for later use: Let α_1 and α_2 be two rational numbers such that

$$0 < \alpha_1 < \alpha \quad \text{and} \quad \alpha_1 + \alpha_2 = \alpha. \tag{9}$$

Let M be a natural number such that $M > \max\{\alpha + \sigma, 1\}$ and $\max\{|\lambda_j|, |\mu_l| : 1 \le j \le k, k+1 \le l \le n\} \le M - 1$. We now construct two simple piecewise-smooth close curves Γ_1 and Γ_2 in $\mathbb{R}^2$: Γ_1 is the boundary of the rectangle with the vertices $(-\alpha - \sigma, M)$, $(-M, M)$, $(-M, -M)$, and $(-\alpha - \sigma, -M)$, while Γ_2 is the boundary of the rectangle with the vertices (σ, M), (M, M), $(M, -M)$, and $(\sigma, -M)$. Then Γ_1 with positive direction (counterclockwise) encloses in its interior all the λ_j for $1 \le j \le k$ and Γ_2 with positive direction encloses all the μ_j for $k+1 \le j \le n$ in its interior. We observe that for any $\xi \in \Gamma_1 \bigcup \Gamma_2$, the matrix $A - \xi I_n$ is invertible. Since the function $g : \Gamma_1 \bigcup \Gamma_2 \to \mathbb{R}$, $g(\xi) = ||(A - \xi I_n)^{-1}||$, is computable (see for example [Zho09]), where $(A - \xi I_n)^{-1}$ is the inverse of the matrix $A - \xi I_n$, the maximum of g on $\Gamma_1 \bigcup \Gamma_2$ is computable. Let $K_1 \in \mathbb{N}$ be an upper bound of this computable maximum. Now for any $t \in \mathbb{R}$, from (5.47) of [Kat95],

$$\begin{aligned} e^{At} &= -\frac{1}{2\pi i} \int_{\Gamma_1} e^{\xi t} (A - \xi I_n)^{-1} d\xi - \frac{1}{2\pi i} \int_{\Gamma_2} e^{\xi t} (A - \xi I_n)^{-1} d\xi \\ &= I_{\Gamma_1}(t) + I_{\Gamma_2}(t) \end{aligned} \tag{10}$$

We recall that e^{At} is the solution to the linear equation $\dot{x} = Ax$. Since A is computable and integration is a computable operator, it follows that I_{Γ_1} and I_{Γ_2} are computable. A simple calculation shows that $||-\frac{1}{2\pi i} \int_{\Gamma_1} e^{t\xi} (A - \xi I_n)^{-1} d\xi|| \le 4K_1 M e^{-(\alpha+\sigma)t}/\pi$ for $t \ge 0$ and $|| - \frac{1}{2\pi i} \int_{\Gamma_2} e^{t\xi} (A - \xi I_n)^{-1} d\xi|| \le 4K_1 M e^{\sigma t}/\pi$ for $t \le 0$. Let

$$K = 4MK_1 \tag{11}$$

Then

$$||I_{\Gamma_1}(t)|| \le K e^{-(\alpha+\sigma)t} \text{ for } t \ge 0 \;\text{ and }\; ||I_{\Gamma_2}(t)|| \le K e^{\sigma t} \text{ for } t \le 0 \tag{12}$$

The two estimates in (12) show that $I_{\Gamma_1}(t)$ and $I_{\Gamma_2}(t)$ are stable component and unstable component of e^A. The first step is now complete.

The second step in the construction is to compute a ball, $B(0, r)$, surrounding the hyperbolic equilibrium point 0 such that $B(0, r)$ contains a set of potential solutions $x(t, x_0)$ to (7) satisfying the conditions (i) and (ii) described in Theorem 9. The details for step 2:

Claim 1. Consider the integral equation

$$u(t,a) = I_{\Gamma_1}(t)a + \int_0^t I_{\Gamma_1}(t-s)F(u(s,a))ds - \int_t^\infty I_{\Gamma_2}(t-s)F(u(s,a))ds \quad (13)$$

where the constant vector a is a parameter. If $u(t,a)$ is a continuous solution to the integral equation, then it satisfies the differential equation (7) with the initial condition $u(0,a) = I_{\Gamma_1}(0)a - \int_0^\infty I_{\Gamma_2}(-s)F(u(s,a))ds$.

Proof of Claim 1. See Appendix 1.

We observe that the solution to the integral equation is the fixed point of the operator defined on the right hand side of the equation (13). Next we compute an integer m_0 such that $2^{-m_0} \le \frac{\sigma}{4K}$, and then set

$$r = 2^{-d(m_0)}/2K, \qquad B = B(0,r) = \{a \in \mathbb{R}^n : |a| < r\} \quad (14)$$

where the computable function d is as in (8).

Claim 2. For any $a \in B$ and $t \ge 0$, define the successive approximations as follows:

$$u^{(0)}(t,a) = 0 \quad (15)$$

$$u^{(j)}(t,a) = I_{\Gamma_1}(t)a + \int_0^t I_{\Gamma_1}(t-s)F(u^{(j-1)}(s,a))ds$$
$$- \int_t^\infty I_{\Gamma_2}(t-s)F(u^{(j-1)}(s,a))ds, \quad j \ge 1$$

then the following three inequalities hold for all $j \in \mathbb{N}$:

$$|u^{(j)}(t,a) - u^{(j-1)}(t,a)| \le K|a|e^{-\alpha_1 t}/2^{j-1} \quad (16)$$

$$|u^{(j)}(t,a)| \le 2^{-d(m_0)}e^{-\alpha_1 t} \quad (17)$$

and for any $\tilde{a} \in B$,

$$|u^{(j)}(t,a) - u^{(j)}(t,\tilde{a})| \le 3K|a - \tilde{a}| \quad (18)$$

Proof of Claim 2. See Appendix 2.

It then follows from (15) and (16) that $\{u^{(j)}(t,a)\}_{j=1}^\infty$ is a computable Cauchy sequence effectively convergent to the solution $u(t,a)$ of the integral equation (13), uniformly in $t \ge 0$ and $a \in B$. Consequently the solution, $t, a \mapsto u(t,a)$, is computable. Furthermore, (17) and (18) imply that for all $t \ge 0$ and $a, \tilde{a} \in B$,

$$|u(t,a)| \le 2^{-d(m_0)}e^{-\alpha_1 t}, \qquad |u(t,a) - u(t,\tilde{a}| < 3K|a - \tilde{a}| \quad (19)$$

Thus $\lim_{t\to\infty} u(t,a) = 0$ for all $a \in B$. Moreover, the first inequality in (19) shows that $u(t,a)$ satisfies condition (ii) of Theorem 9 with $\gamma = 2^{-d(m_0)}$, $\epsilon = \alpha_1$, and $\delta = 2^{-d(m_0)}/2K$.

Although Claim 2 shows that for any $a \in B$, the integral equation (13) has a computable solution $u(t,a)$ and this solution satisfies the conditions (i)

and (ii) of Theorem 9, we may not take B as a desired stable manifold S because if $u(0,a) \neq a$, then $u(t,a)$ is not a solution to the equation (6) with the initial value a at $t = 0$. Nevertheless, the set B provides a pool of potential solutions to (6) on a stable manifold. This leads us to the next step of the proof.

The last step in the construction of the desired algorithm is to extract a k-dimensional manifold S from B, using a computable process, such that for any $a \in S$, $u(0,a) = a$. Then if we set $x(t,a) = u(t,a)$ for $a \in S$, by Claims 1 and 2, $x(t,a)$ is the solution to the differential equation (7) with the initial value a at $t = 0$ and satisfies the conditions (i) and (ii) of Theorem 9. Now for the details.

Let $P_1 = I_{\Gamma_1}(0) = -\frac{1}{2\pi i}\int_{\Gamma_1}(A-\xi I_n)^{-1}d\xi$ and $P_2 = I_{\Gamma_2}(0) = -\frac{1}{2\pi i}\int_{\Gamma_2}(A-\xi I_n)^{-1}d\xi$. Then $P_1\mathbb{R}^n = \mathbb{E}^s_A$, $P_2\mathbb{R}^n = \mathbb{E}^u_A$, $\mathbb{R}^n = \mathbb{E}^s_A \bigoplus \mathbb{E}^u_A$ with $\dim\mathbb{E}^s_A = k$ and $\dim\mathbb{E}^u_A = n-k$, $P_jP_k = \delta_{jk}P_j$ ($\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$), and $P_1 + P_2 = I$ is the identity map on $\mathbb{R}^n$ (*c.f.* §1.5.3 and §1.5.4, [Kat95]; §4.6 of [Rob95]). Since A is computable, so are P_1 and P_2. Moreover, $I_{\Gamma_1}(t)P_2 = 0$ for any $t \in \mathbb{R}$ as the following calculation shows: Let $R(\xi)$ denote $(A-\xi I_n)^{-1}$. Then we have

$$\begin{aligned}
R(\xi_1) - R(\xi_2) &= R(\xi_1)(A-\xi_2 I_n)R(\xi_2) - R(\xi_1)(A-\xi_1 I_n)R(\xi_2)\\
&= R(\xi_1)[(A-\xi_2 I_n) - (A-\xi_1 I_n)]R(\xi_2)\\
&= R(\xi_1)(\xi_1-\xi_2)I_n R(\xi_2)\\
&= (\xi_1-\xi_2)R(\xi_1)R(\xi_2).
\end{aligned}$$

Using the last equation we obtain that for any $v \in \mathbb{R}^n$,

$$\begin{aligned}
I_{\Gamma_1}(t)P_2 v &= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1} e^{\xi t}R(\xi)d\xi \int_{\Gamma_2} R(\xi')vd\xi' \qquad (20)\\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} e^{\xi t}R(\xi)R(\xi')vd\xi d\xi'\\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1}\int_{\Gamma_2} e^{\xi t}\frac{R(\xi)-R(\xi')}{\xi-\xi'}vd\xi d\xi'\\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1} e^{\xi t}\left(\int_{\Gamma_2}\frac{R(\xi)}{\xi-\xi'}d\xi' - \int_{\Gamma_2}\frac{R(\xi')}{\xi-\xi'}d\xi'\right)vd\xi\\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_1} e^{\xi t}\left(-\int_{\Gamma_2}\frac{R(\xi')}{\xi-\xi'}d\xi'\right)vd\xi\\
&= \left(\frac{1}{2\pi i}\right)^2 \int_{\Gamma_2} R(\xi')\left(\int_{\Gamma_1}\frac{e^{\xi t}}{\xi'-\xi}d\xi\right)vd\xi' = 0
\end{aligned}$$

A similar computation shows that for any $t \in \mathbb{R}$,

$$I_{\Gamma_2}(t)P_1 v = 0 \quad \text{and} \quad P_2 I_{\Gamma_2}(t)v = I_{\Gamma_2}(t)v, \quad v \in \mathbb{R}^n \qquad (21)$$

Now let us use these results to compute the projections of $u(0,a)$ in $\mathbb{E}^s_A$ and $\mathbb{E}^u_A$:

for any $a \in \mathbb{R}^n$,

$$\begin{aligned} P_1 u(0,a) &= P_1 \left(I_{\Gamma_1}(0)a - \int_0^\infty I_{\Gamma_2}(-s)F(u(s,a))ds \right) \\ &= P_1 P_1 a - \int_0^\infty P_1 I_{\Gamma_2}(-s)F(u(s,a))ds \\ &= P_1 a \end{aligned} \tag{22}$$

and

$$\begin{aligned} P_2 u(0,a) &= P_2 \left(I_{\Gamma_1}(0)a - \int_0^\infty I_{\Gamma_2}(-s)F(u(s,a))ds \right) \\ &= P_2 P_1 a - P_2 \left(\int_0^\infty I_{\Gamma_2}(-s)F(u(s,a))ds \right) \\ &= - \int_0^\infty I_{\Gamma_2}(-s)F(u(s,a))ds \end{aligned}$$

We note that for any $a \in \mathbb{R}^n$,

$$\begin{aligned} I_{\Gamma_1}(t)a &= I_{\Gamma_1}(t)(P_1 a + P_2 a) \\ &= I_{\Gamma_1}(t)P_1 a + I_{\Gamma_1}(t)P_2 a \\ &= I_{\Gamma_1}(t)P_1 a \end{aligned}$$

which implies that if the solution $u(t,a)$ of the integral equation (13) is constructed by successive approximations (15), then $P_2 a$ does not enter the computation for $u(t,a)$ and thus may be taken as zero. Therefore the projection of $u(0,a)$ in $\mathbb{E}^u_A$ satisfies the equation

$$P_2 u(0,a) = - \int_0^\infty I_{\Gamma_2}(-s)F(u(s,P_1 a))ds \tag{23}$$

Next we define a map $\phi_A : \mathbb{E}^s_A(r) \to \mathbb{E}^u_A$, $b \mapsto -\int_0^\infty I_{\Gamma_2}(-s)F(u(s,b))ds$ for $b \in \mathbb{E}^s_A(r)$, where r is the rational number defined in (14) and $\mathbb{E}^s_A(r) = \{b \in \mathbb{E}^s_A : |b| \le r/2\}$. We observe that the compact set $\mathbb{E}^s_A(r)$ is computable since the closed set $\mathbb{E}^s_A$ is computable (proved in Theorem 8). Obviously the map ϕ_A is computable. By Theorem 6.2.4 [Wei00] the compact set $\phi_A[\mathbb{E}^s_A(r)]$ is computable.

Now we are ready to define S:

$$S = \{b + \phi_A(b) : b \in \mathbb{E}^s_A(r)\}$$

Then S is a manifold of dimension k. For any $a \in S$, $a = b + \phi_A(b)$ for some $b \in \mathbb{E}^s_A(r)$. Since $\mathbb{R}^n = \mathbb{E}^s_A \bigoplus \mathbb{E}^u_A$, $b \in \mathbb{E}^s_A$ and $\phi_A(b) \in \mathbb{E}^u_A$, it follows that

$$b = P_1 a \quad \text{and} \quad \phi_A(b) = \phi_A(P_1 a) = P_2 a \tag{24}$$

Combining (22), (23), and (24) we obtain that for any $a \in S$, $u(0,a) = P_1 u(0,a) + P_2 u(0,a) = P_1 a + \phi_A(P_1 a) = P_1 a + P_2 a = a$. Thus for any $a \in S$, $u(0,a) = a$; that is, a is the initial condition of $u(t,a)$ at $t = 0$.

Claim 3. S is a computable closed subset of $\mathbb{R}^n$.

Proof. Since $\mathbb{E}^s_A(r)$ is computable, there is a computable sequence $\{b_j\} \subseteq \mathbb{E}^s_A(r)$ that is effectively dense in $\mathbb{E}^s_A(r)$; that is, there is a computable function $\psi : \mathbb{N} \to \mathbb{N}$ such that $\mathbb{E}^s_A(r) \subseteq \bigcup_{j=1}^{\psi(k)} B(b_j, 2^{-k})$ for all $k \in \mathbb{N}$ (*c.f.* [Zho96]). The following estimate shows that the computable sequence $\{b_j + \phi_A(b_j)\}$ is effectively dense in S, thus the closed manifold S is computable. For any $b+\phi_A(b), \tilde{b}+\phi_A(\tilde{b}) \in S$, we have

$$\begin{aligned}
&|b + \phi_A(b) - (\tilde{b} + \phi_A(\tilde{b}))| \\
&\le |b - \tilde{b}| + |\phi_A(b) - \phi_A(\tilde{b})| \\
&= |b - \tilde{b}| + \left| -\int_0^\infty I_{\Gamma_2}(-s)F(u(s,b))ds + \int_0^\infty I_{\Gamma_2}(-s)F(u(s,\tilde{b}))ds \right| \\
&\le |b - \tilde{b}| + \int_0^\infty ||I_{\Gamma_2}(-s)|| \cdot |F(u(s,b)) - F(u(s,\tilde{b}))|ds \\
&\le |b - \tilde{b}| + 2^{-m_0}|u(s,b) - u(s,\tilde{b})| \int_0^\infty Ke^{-\sigma s}ds \\
&= |b - \tilde{b}| + \frac{K}{\sigma 2^{m_0}}|u(s,b) - u(s,\tilde{b})| \\
&= |b - \tilde{b}| + \frac{K}{\sigma 2^{m_0}} \cdot 3K|b - \tilde{b}|
\end{aligned}$$

The estimates (12), (8), (14), and (18) are used in the above calculation. The proof of claim 3 is complete.

For every $a \in S$, set $x(t,a) = u(t,a)$; then by Claims 1, 2, and 3, $x(t,a)$ is the solution to the differential equation (7) with initial value a at $t = 0$ and $x(t,a)$ satisfies the conditions (i) and (ii) of Theorem 9.

Finally we show that if $x(t,x_0)$ is a solution to the differential equation (7) satisfying $0 < |x_0| < 2^{-d(m_0)}/4K^2$ but $x_0 \notin S$, then there exists $t' > 0$ such that $|x(t',x_0)| > 2^{-d(m_0)}$. This proves the last part of the theorem if we set $\eta = 2^{-d(m_0)}$ and $D = \{x \in \mathbb{R}^n : |x| < 2^{-d(m_0)}/4K^2\}$. Indeed, if otherwise $|x(t,x_0)| \le 2^{-d(m_0)}$ for all $t \ge 0$. We show in the following that this condition implies $x_0 \in S$, which is a contradiction.

Since $x(t,x_0)$ is the solution to $\dot{x} = Ax + F(x)$ with the initial value x_0, it follows that (*c.f.* Theorem 4.8.2 [Rob95])

$$x(t,x_0) = e^{At}x_0 + \int_0^t e^{(t-s)A}F(x(s,x_0))ds$$

which, using (10), can be rewritten as

$$\begin{aligned}
&x(t,x_0) \qquad\qquad (25)\\
&= I_{\Gamma_1}(t)x_0 + I_{\Gamma_2}(t)x_0 + \int_0^t I_{\Gamma_1}(t-s)F(x(s,x_0))ds + \int_0^t I_{\Gamma_2}(t-s)F(x(s,x_0))ds \\
&= I_{\Gamma_1}(t)x_0 + \int_0^t I_{\Gamma_1}(t-s)F(x(s,x_0))ds - \int_t^\infty I_{\Gamma_2}(t-s)F(x(s,x_0))ds + I_{\Gamma_2}(t)b
\end{aligned}$$

where $b = x_0 + \int_0^\infty I_{\Gamma_2}(-s)F(x(s,x_0))ds$ (*c.f.* §1.5.3 [Kat95]). Note that b is well defined since we assume that $|x(t,x_0)| \le 2^{-d(m_0)}$ for all $t \ge 0$, then, from (8), $|F(x(t,x_0))|$ is bounded for all $t \ge 0$; consequently, by (12), the integral $\int_0^\infty I_{\Gamma_2}(-s)F(x(s,x_0))ds$ converges. We also note that the first three

terms in the above representation for $x(t, x_0)$ are bounded. Moreover, $I_{\Gamma_2}(t)b = I_{\Gamma_2}(t)P_1b + I_{\Gamma_2}(t)P_2b = I_{\Gamma_2}(t)P_2b$ since $I_{\Gamma_2}(t)P_1b = 0$ by (21). We claim that if $P_2b \not\equiv 0$, then $I_{\Gamma_2}(t)b$ is unbounded as $t \to \infty$. We make use of the residue formula to prove the claim. Recall that Γ_2 is a closed curve in the right-hand side of the complex plane with counterclockwise orientation that contains in its interior μ_j, $k+1 \leq j \leq n$, where μ_j are the eigenvalues of A with $Re(\mu_j) > 0$, which are the exact singularities of $R(\xi) = (A - \xi I_n)^{-1}$ in the right complex plane. Then by the residue formula,

$$\frac{1}{2\pi i}\int_{\Gamma_2} e^{t\xi}R(\xi)d\xi = \sum_{l=k+1}^{n} e^{\mu_j t}\mathrm{res}_{\mu_j}R \tag{26}$$

where $\mathrm{res}_{\mu_j}R$ is the residue of R at μ_j. Since $Re(\mu_j) > 0$, if $P_2b \not\equiv 0$, then

$$\begin{aligned} I_{\Gamma_2}(t)b &= I_{\Gamma_2}(t)P_2b \\ &= -\frac{1}{2\pi i}\int_{\Gamma_2} e^{t\xi}(A - \xi I_n)^{-1}P_2bd\xi \\ &= -\sum_{l=k+1}^{n} e^{\mu_j t} res_{\mu_j}RP_2b \end{aligned}$$

is unbounded as $t \to \infty$. This is however impossible because the first three terms in (25) are bounded and we have assumed that $|x(t, x_0)| \leq 2^{-d(m_0)}$ for all $t \geq 0$. Therefore, $P_2b \equiv 0$; consequently, $I_{\Gamma_2}(t)b = I_{\Gamma_2}(t)P_2b = 0$. This last equation together with (25) shows that $x(t, x_0)$ satisfies the integral equation (13). Now let $x'(t, x_0)$ be the solution to the integral equation (13) with parameter x_0 and constructed by the successive approximations (15). Then

$$x'(t, x_0) = I_{\Gamma_1}(t)x_0 + \int_0^t I_{\Gamma_1}(t-s)F(x'(s, x_0))ds - \int_t^\infty I_{\Gamma_2}(t-s)F(x'(s, P_1x_0))ds$$

By the uniqueness of the solution, $x(t, x_0) = x'(t, x_0)$; in particular, $x_0 = x(0, x_0) = x'(0, x_0) = P_1x_0 - \int_0^\infty I_{\Gamma_2}(-s)F(x'(s, P_1x_0))ds$. Since $P_1x_0 \in \mathbb{E}_A^s$ and $\|P_1x_0\| \leq \|P_1\|\|x_0\| \leq K \cdot 2^{d(m_0)}/4K^2 = r/2$ (recall that $\|P_1\| \leq K$ by (12)), it follows that $P_1x_0 \in \mathbb{E}_A^s(r)$, which further implies that $x_0 = P_1x_0 - \int_0^\infty I_{\Gamma_2}(-s)F(x'(s, P_1x_0))ds = P_1x_0 + \phi_A(P_1x_0) \in S$. This contradicts the fact that x_0 is not on S. The proof is complete. ■

Theorem 10 *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^1-computable function. Assume that the origin 0 is a hyperbolic equilibrium point of (6) such that $Df(0)$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part (counting multiplicity), $0 \leq k < n$. Let $x(t, x_0)$ denote the solution to (6) with the initial value x_0 at $t = 0$. Then there is a (Turing) algorithm that computes a $(n-k)$-dimensional manifold $U \subset \mathbb{R}^n$ containing 0 such that*

(i) For all $x_0 \in U$, $\lim_{t\to-\infty} x(t, x_0) = 0$;

(ii) There are three positive rational numbers γ, ϵ, and δ such that $|x(t, x_0)| \leq \gamma 2^{\epsilon t}$ for all $t \leq 0$ whenever $x_0 \in U$ and $|x_0| \leq \delta$.

Moreover, if $k > 0$, then a rational number η and a ball D can be computed from f such that for any solution $x(t, x_0)$ to the equation (6) with $x_0 \in D \setminus U$, $\{x(t, x_0) : t \leq 0\} \not\subset B(0, \eta)$ no matter how close the initial value x_0 is to the origin.

Proof. The unstable manifold U can be computed by the same procedure as the construction of S by considering the equation

$$\dot{x} = -Ax - F(x(t))$$

■

The proof can be easily extended to show that the map: $\mathfrak{F}_H \to \mathfrak{K} \times \mathfrak{K}$, $f \mapsto (S_f, U_f)$, is computable, where $\mathfrak{F}_H$ is the set of all C^1 functions having the origin as a hyperbolic equilibrium point, $\mathfrak{K}$ is the set of all compact subsets of $\mathbb{R}^n$, and S_f and U_f are some local stable and unstable manifolds of f at the origin respectively.

5 Computability and non-computability of global stable/unstable manifolds

Although the stable and unstable manifolds can be computed locally as shown in the previous section, globally they may not be necessarily computable.

In [Zho09] it is shown that there exists a C^∞ and polynomial-time computable function $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that the equation $\dot{x} = f(x)$ has a sink at the origin $\mathbf{0}$ and the basin of attraction (also called the domain of attraction), $B_f(\mathbf{0})$, of f at $\mathbf{0}$ is a non-computable open subset of $\mathbb{R}^2$. Since a sink is a hyperbolic equilibrium point (with all eigenvalues of the gradient matrix $Df(\mathbf{0})$ having negative real parts) and the basin of attraction at a sink is exactly the global stable manifold at this equilibrium point, we conclude immediately that the global stable manifold is not necessarily computable.

One the other hand, it is also shown in [Zho09] that, although not necessarily computable, the basin of attraction $B_f(x_0)$ is r.e. open for a computable sink x_0 of a computable system f. In other words, the open subset $B_f(x_0)$ can be plotted from inside by a computable sequence $\{B_n\}$ of rational open balls, $B_f(x_0) = \bigcup B_n$, but one may not know the rate at which $B_f(x_0)$ is being filled up by B_n's if $B_f(x_0)$ is non-computable. When an equilibrium point x_0 of the system $\dot{x} = f(x)$ is a saddle (not a sink nor a source), then the global stable (unstable) manifold $W_f^s(x_0)$ ($W_f^u(x_0)$) is in general an F_σ-set of $\mathbb{R}^n$. As in the special case of the basin of attraction, $W_f^s(x_0)$ can also be plotted from inside by a computable process. To make the concept precise, we introduce the following definition. Let $\mathcal{F}(\mathbb{R}^n)$ denote the set of all F_σ-subsets of $\mathbb{R}^n$.

Definition 11 *A function $f : X \to \mathcal{F}(\mathbb{R}^n)$ is called semi-computable if there is a Type-2 machine such that on any ρ-name of $x \in X$, the machine computes as output a sequence $\{a_{j,k}\}$, $a_{j,k} \in \mathbb{Q}^n$, such that*

$$f(x) = \bigcup_{j=0}^{\infty} \overline{\{a_{j,k} : k \in \mathbb{N}\}}$$

where $\overline{A}$ denotes the closure of the set A.

We call this function semi-computable because we can plot a dense subset of the set $f(x)$ by a computable process, but we cannot tell in a finite amount of time what the "density" is.

Theorem 12 *The map* $\psi : C^1(\mathbb{R}^n) \times \mathbb{R}^n \to \mathcal{F}(\mathbb{R}^n)$, $(f, x_0) \to W_f^s(x_0)$, *is semi-computable, where* $(f, x_0) \in dom(\psi)$ *if* x_0 *is a hyperbolic equilibrium point of* f, *and* $W_f^s(x_0)$ *is the global stable manifold of* f *at* x_0.

Proof. From Theorem 9 one can compute from f and x_0 a compact subset S of $\mathbb{R}^n$, which is a local stable manifold of f at x_0. Note that the global stable manifold of f at x_0 is the union of the backward flows of S, *i.e.*,

$$W_f^s(x_0) = \bigcup_{j=0}^{\infty} \phi_{-j}(S)$$

where $\phi_t(a)$ is the flow induced by the equation $\dot{x} = f(x)$ at time t with the initial data $x(0) = a$. Since the sequence $\{\psi_{-j}(a)\}$ is computable from f and a [GZB09], it follows that the sequence $\{\phi_{-j}(S)\}$ of compact sets are computable from f and x_0 (Theorem 6.2.4 [Wei00]). In particular, a sequence $\{a_{j,k}\}_{j,k\in\mathbb{N}} \subset \mathbb{R}^n$ can be computed such that $W_f^s(x_0) = \bigcup_{j=0}^{\infty} \overline{\{a_{j,k} : k \in \mathbb{N}\}}$. ■

The function f in the counterexample mentioned at the beginning of this section is C^∞ but not analytic. It is an open problem whether or not the stable/unstable manifold(s) of a computable analytic hyperbolic system $\dot{x} = f(x)$ (*i.e.* f is computable and analytic) is computable. In the following we present a negative answer to a weaker version of the open problem; we show that it is impossible to compute uniformly the closure of the global stable/unstable manifold from (f, x_0), where f is analytic and x_0 is a hyperbolic equilibrium point of f. Let's denote by $\omega(\mathbb{R}^n)$ the set of real analytic functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathcal{A}(\mathbb{R}^n)$ the set of all closed subsets of $\mathbb{R}^n$. With Wijsman topology $\mathcal{W}$ on $\mathcal{A}(\mathbb{R}^n)$, $(\mathcal{A}(\mathbb{R}^n), \mathcal{W})$ is a topological space that is separable and metrizable with a complete metric. It can be shown that the Wijsman topology on $\mathcal{A}(\mathbb{R}^n)$ is the same as the topology induced by the ρ-names giving rise to the notion of computable close subsets of $\mathbb{R}^n$ as defined in Def. 6 ([BW99]). Thus if a map $F : X \to \mathcal{A}(\mathbb{R}^n)$ is computable (with respect to above ρ-names for elements of $\mathcal{A}(\mathbb{R}^n)$), then F is continuous (with respect to Wijsman topology on $\mathcal{A}(\mathbb{R}^n)$) (Corollary 3.2.12 of [Wei00]).

Theorem 13 *The map* $\psi : \omega(\mathbb{R}^n) \times \mathbb{R}^n \to \mathcal{A}(\mathbb{R}^n)$, $(f, x_0) \to \overline{W_f^u(x_0)}$ *(the closure of* $W_f^u(x_0)$*), is not computable, where* $(f, x_0) \in dom(\psi)$ *if* x_0 *is a hyperbolic equilibrium point of* f, *and* $W_f^u(x_0)$ *is the global unstable manifold of* f *at* x_0.

Proof. Consider the following system $\dot{x} = f_\mu(x)$ taken from [HSD04],

$$\begin{aligned} x' &= x^2 - 1 \\ y' &= -xy + \mu(x^2 - 1) \end{aligned} \tag{27}$$

The system (27) has two equilibria: $z_1 = (-1, 0)$ and $z_2 = (1, 0)$. $Df_\mu(z_1)$ has eigenvalues -2 and 1, associated to the eigenvectors $(-1, -2\mu/3)$ and $(0, 1)$, respectively, and $Df_\mu(z_2)$ has eigenvalues 2 and -1, associated to the eigenvectors $(-1, -2\mu/3)$ and $(0, 1)$, respectively. Therefore both points z_1 and z_2 are saddles (the behavior of the system is sketched in Fig. 2). From this information and the fact that any point $(-1, y)$ or $(1, y)$ can only move along the y axis, one concludes that the line $x = -1$ is the unstable manifold of z_1 and the line $x = 1$ gives the stable manifold of z_2. Note that z_1, z_2 and the above manifolds do not depend on μ.

Let us now study the unstable manifold of z_2. It is split by the stable manifold $x = 1$, and hence there is a "right" as well as a "left" portion of the unstable manifold. Let us focus our attention to the "left" portion. From the system (27) one concludes that any point z near z_2, located to its left (i.e. $z < z_2$) will be pushed to the line $x = -1$ at a rate that is independent of the y-coordinate of z.

When $\mu = 0$, the eigenvector of $Df_\mu(z_2)$ associated to the unstable manifold is $(-1, 0)$. Looking at (27), one concludes that the "left" portion of the unstable manifold of z_2 can only be the segment of line

$$U_{l,0} = \{(x, 0) \in \mathbb{R}^2 | -1 < x < 1\}.$$

Now let us analyze the case where $\mu < 0$. Since $(-1, -2\mu/3)$ is the eigenvector of $Df_\mu(z_2)$ associated to the unstable manifold, as the unstable manifold of z_2 moves to the left, its y-coordinate starts to grow. The "left" portion of the unstable manifold is always above the line $y = 0$ (if it could be $y = 0$, the dynamics of (27) would push the trajectory upwards), and as soon as its x-coordinate is less than 0 (this will eventually happen), the trajectory is pushed upwards with y-coordinate converging to $+\infty$. Notice that the closer μ is to 0, the lesser the y-component of the trajectory grows, and the closer to z_1 the unstable manifold will be.

If $U_{l,\mu}$ represents the left portion of the unstable manifold of z_2 and

$$A = U_{l,0} \cup \{(-1, y) \in \mathbb{R}^2 | 0 \le y\}$$

then one concludes that

$$\lim_{\mu \to 0^-} d(A, U_{l,\mu}) = 0$$

where d is the Hausdorff distance on $\mathbb{R}^n$.

Suppose that ψ is computable. Then, in particular, the map $\chi : \mathbb{R} \to \mathcal{A}(\mathbb{R}^n)$ defined by $\chi(\mu) = \overline{W^u_{f_\mu}(z_2)}$, is also computable. Since computable maps must be continuous (Corollary 3.2.12 of [Wei00]), this implies that χ should be a continuous map. But the point $\mu = 0$ is a point of discontinuity for χ since (without loss of generality, we restrict ourselves to the semi-plane $x < 1$)

$$\lim_{\mu \to 0^-} \chi(\mu) = \lim_{\mu \to 0^-} \overline{U}_{l,\mu} = \overline{A} \neq \overline{U}_{l,0} = \chi(0).$$

and hence this map cannot be computable, which implies that ψ is not computable either. ■

6 The Smale horseshoe is computable

In this section we show that Smale's horseshoe is computable.

Theorem 14 *The Smale Horseshoe Λ is a computable (recursive) closed subset in $I = [0, 1] \times [0, 1]$.*

Proof. We show that $\Omega = I \setminus \Lambda$ is a computable open subset in I by making use of the fact [Zho96]: An open subset $U \subseteq I$ is computable if and only if there is a computable sequence of rational open rectangles (having rational corner points) in I, $\{J_k\}_{k=0}^{\infty}$, such that

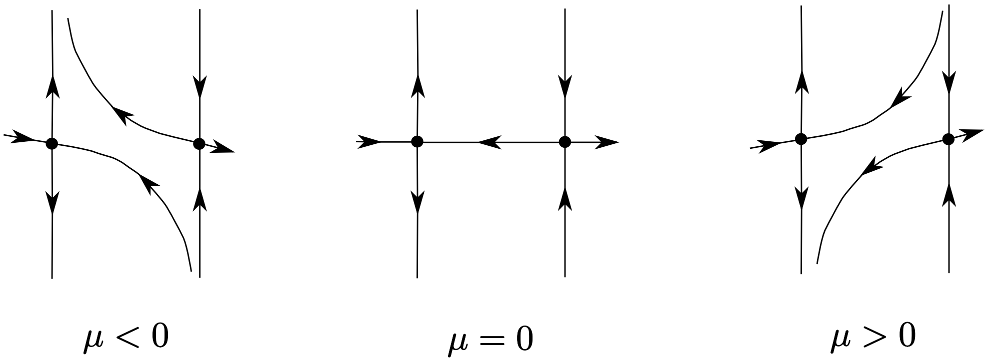

Figure 2: Dynamics of the system (27).

(a) $J_k \subset U$ for all $k \in \mathbb{N}$,

(b) the closure of J_k, $\bar{J}_k$, is contained in U for all $k \in \mathbb{N}$, and

(c) there is a recursive function $e : \mathbb{N} \to \mathbb{N}$ such that the Hausdorff distance $d(I \setminus \cup_{k=0}^{e(n)} J_k, I \setminus U) \leq 2^{-n}$ for all $n \in \mathbb{N}$.

Let $f : I \to \mathbb{R}^2$ be a map such that $\Lambda = \bigcap_{n=-\infty}^{\infty}(f^n(I) \cap I)$ is the Smale horseshoe. Without loss of generality assume that f performs a linear vertical expansion by a factor of $\mu = 4$ and a linear horizontal contraction by a factor of $\lambda = \frac{1}{4}$. For each $n \in \mathbb{N}$, let $U_n = I \setminus \bigcap_{k=-n}^{n}(f^k(I) \cap I)$. Then $I \setminus \Lambda = \bigcup_{n=0}^{\infty} U_n$. Moreover,

(1) there exists a computable function $\alpha : \mathbb{N} \to \mathbb{N}$ such that U_n is a union of $\alpha(n)$ rational open rectangles in I,

(2) $d(\Lambda, I \setminus \bigcup_{k=0}^{n} U_n) \leq (1/4)^n$.

Define $e : \mathbb{N} \to \mathbb{N}$, $e(n) = \sum_{k=0}^{n} \alpha(k)$, $n \in \mathbb{N}$. Then it follows from the lemma that $I \setminus \Lambda$ is indeed a computable open subset of I. ∎

7 Conclusions

We have shown that, locally, one can compute the stable and unstable manifolds of some given hyperbolic equilibrium point, though globally these manifolds are, in general, only semi-computable. It would be interesting to know if these results are only valid to equilibrium points or can be extended e.g. to hyperbolic periodic orbits. In [GZar] we provide an example which shows that the global stable/unstable manifold cannot be computed for hyperbolic periodic orbits. However, the question whether locally these manifolds can be computed remains open.

Appendix 1. Proof of Claim 1. Assume that $u(t, a)$ is a continuous solution to the integral equation (13). We show that it satisfies the differential equation

(7). To see this, we first establish a relation between I_{Γ_j}, $j = 1, 2$, and the Jordan canonical form of A:

$$A = C \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} C^{-1}$$

where C is an $n \times n$ invertible matrix, P is a $k \times k$ matrix with eigenvalues λ_j, $1 \le j \le k$, and Q is a $(n-k) \times (n-k)$ matrix with eigenvalues μ_j, $k+1 \le j \le n$. By (10), we have

$$e^{At} = I_{\Gamma_1}(t) + I_{\Gamma_2}(t)$$

On the other hand, it is straightforward to show that

$$e^{At} = C \begin{pmatrix} e^{Pt} & 0 \\ 0 & 0 \end{pmatrix} C^{-1} + C \begin{pmatrix} 0 & 0 \\ 0 & e^{Qt} \end{pmatrix} C^{-1}$$

Since the first k columns of C consists of a basis of the stable subspace $P_1\mathbb{R}^n$ and the last $(n-k)$ columns of C is a basis of the unstable subspace $P_2\mathbb{R}^n$, it then follows from (20) and (21) that

$$I_{\Gamma_2}(t) C \begin{pmatrix} e^{Pt} & 0 \\ 0 & 0 \end{pmatrix} C^{-1} = 0, \quad \text{and}$$

$$I_{\Gamma_1}(t) C \begin{pmatrix} 0 & 0 \\ 0 & e^{Qt} \end{pmatrix} C^{-1} = 0$$

Thus

$$I_{\Gamma_1}(t) = C \begin{pmatrix} e^{Pt} & 0 \\ 0 & 0 \end{pmatrix} C^{-1}$$

and

$$I_{\Gamma_2}(t) = C \begin{pmatrix} 0 & 0 \\ 0 & e^{Qt} \end{pmatrix} C^{-1}$$

(recall that $\mathbb{R}^n = P_1\mathbb{R}^n \bigoplus P_2\mathbb{R}^n$). Consequently the integral equation (13) can be written in the following form:

$$\begin{aligned} u(t, a) = {} & C \begin{pmatrix} e^{Pt} & 0 \\ 0 & 0 \end{pmatrix} C^{-1} a + \int_0^t C \begin{pmatrix} e^{P(t-s)} & 0 \\ 0 & 0 \end{pmatrix} C^{-1} F(u(s, a)) ds \qquad (28) \\ & - \int_t^\infty C \begin{pmatrix} 0 & 0 \\ 0 & e^{Q(t-s)} \end{pmatrix} C^{-1} F(u(s, a)) ds \end{aligned}$$

It is known that if $u(t, a)$ is a continuous solution to (28), then it is the solution to (7) (*c.f.* §2.7 [Per01]). The proof is complete.

Appendix 2. Proof of Claim 2. For $a, \tilde{a} \in B$ and $t \ge 0$, where $B = B(0, r) = \{x \in \mathbb{R}^n : |x| < r\}$, $r = 2^{-d(m_0)}/2K$ (d is the computable function defined in (8), and m_0 is a positive integer such that $2^{-m_0} \le \frac{\sigma}{4K}$, define

$$\begin{aligned} u^{(0)}(t, a) &= 0 \\ u^{(j)}(t, a) &= I_{\Gamma_1}(t) a + \int_0^t I_{\Gamma_1}(t-s) F(u^{(j-1)}(s, a)) ds \\ &\quad - \int_t^\infty I_{\Gamma_2}(t-s) F(u^{(j-1)}(s, a)) ds, \quad j \ge 1 \end{aligned}$$

We show that the following three inequalities hold for all $j \in \mathbb{N}$:

$$|u^{(j)}(t,a) - u^{(j-1)}(t,a)| \leq K|a|e^{-\alpha_1 t}/2^{j-1}$$
$$|u^{(j)}(t,a)| \leq 2^{-d(m_0)}e^{-\alpha_1 t}$$
$$|u^{(j)}(t,a) - u^{(j)}(t,\tilde{a})| \leq 3K|a-\tilde{a}|$$

We argue by induction on j. Since $u^{(0)}(t,a) = 0$ for any a, by (9) and (12) we get $|u^{(1)}(t,a)| = ||I_{\Gamma_1}(t)a|| \leq Ke^{-(\alpha+\sigma)t} \cdot 2^{-d(m_0)}/2K < 2^{-d(m_0)}e^{-\alpha_1 t}$, and $|u^{(1)}(t,a) - u^{(1)}(t,\tilde{a})| = |I_{\Gamma_1}(t)(a-\tilde{a})| \leq Ke^{-(\alpha+\sigma)t}|a-\tilde{a}| \leq K|a-\tilde{a}|$, the three inequalities hold for $j = 1$. The estimate (12) is used in calculations here.

Assume that the three inequalities hold for all $k \leq j$. Then for $k = j+1$,

$$\begin{aligned}
&|u^{(j+1)}(t,a) - u^{(j)}(t,a)| \\
&\leq \int_0^t ||I_{\Gamma_1}(t-s)|| \cdot |F(u^{(j)}(s,a)) - F(u^{(j-1)}(s,a))|ds \\
&+ \int_t^\infty ||I_{\Gamma_2}(t-s)|| \cdot |F(u^{(j)}(s,a)) - F(u^{(j-1)}(s,a))|ds \\
&\leq \int_0^t ||I_{\Gamma_1}(t-s)|| \cdot 2^{-m_0}|u^{(j)}(s,a) - u^{(j-1)}(s,a)|ds \\
&+ \int_t^\infty ||I_{\Gamma_2}(t-s)|| \cdot 2^{-m_0}|u^{(j)}(s,a) - u^{(j-1)}(s,a)|ds \\
&\leq 2^{-m_0}\int_0^t Ke^{-(\alpha_1+\alpha_2+\sigma)(t-s)}\frac{K|a|e^{-\alpha_1 s}}{2^{j-1}}ds + 2^{-m_0}\int_t^\infty Ke^{\sigma(t-s)}\frac{K|a|e^{-\alpha_1 s}}{2^{j-1}}ds \\
&= 2^{-m_0}\frac{K^2|a|}{2^{j-1}}e^{-(\alpha_1+\alpha_2+\sigma)t}\int_0^t e^{(\alpha_2+\sigma)s}ds + 2^{-m_0}\frac{K^2|a|}{2^{j-1}}e^{\sigma t}\int_t^\infty e^{(-(\alpha_1+\sigma)s}ds \\
&= 2^{-m_0}\frac{K^2|a|}{2^{j-1}}e^{-(\alpha_1+\alpha_2+\sigma)t}\frac{e^{(\alpha_2+\sigma)t}-1}{\alpha_2+\sigma} + 2^{-m_0}\frac{K^2|a|}{2^{j-1}}e^{\sigma t}\frac{0-e^{-(\alpha_1+\sigma)t}}{-(\alpha_1+\sigma)} \\
&\leq 2^{-m_0}\frac{K^2|a|}{2^{j-1}}\frac{e^{-\alpha_1 t}}{\alpha_2+\sigma} + 2^{-m_0}\frac{K^2|a|}{2^{j-1}}\frac{e^{-\alpha_1 t}}{\alpha_1+\sigma} \\
&< \frac{K|a|}{2^j}e^{-\alpha_1 t} \quad \text{(recall that } 2^{-m_0} \leq \tfrac{\sigma}{4K}\text{)}
\end{aligned}$$

and furthermore,

$$\begin{aligned}
&|u^{(j+1)}(t,a)| \\
&\leq |u^{(j)}(t,a)| + |u^{(j+1)}(t,a) - u^{(j)}(t,a)| \\
&\leq |u^{(j)}(t,a)| + \frac{K|a|}{2^j}e^{-\alpha_1 t} \\
&\leq \sum_{k=1}^{j}\frac{K|a|}{2^k}e^{-\alpha_1 t} \qquad \text{(induction hypothesis on } u^{(k)}(t,a) \text{ for } k \leq j\text{)} \\
&\leq 2K|a|e^{-\alpha_1 t} \leq 2Ke^{-\alpha_1 t} \cdot 2^{-d(m_0)}/2K = 2^{-d(m_0)}e^{-\alpha_1 t}
\end{aligned}$$

Lastly we show that if $|u^{(k)}(t,a) - u^{(k)}(t,\tilde{a})| \leq 3K|a-\tilde{a}|$ holds for all $k \leq j$,

then it holds for $j+1$.

$$
\begin{aligned}
&|u^{(j+1)}(t,a) - u^{(j+1)}(t,\tilde{a})| \\
&= \left| I_{\Gamma_1}(t)(a-\tilde{a}) + \int_0^t I_{\Gamma_1}(t-s)\left(F(u^{(j)}(s,a)) - F(u^{(j)}(s,\tilde{a}))\right) ds - \right. \\
&\left. - \int_t^\infty I_{\Gamma_2}(t-s)\left(F(u^{(j)}(s,a)) - F(u^{(j)}(s,\tilde{a}))\right) ds \right| \\
&\le |I_{\Gamma_1}(t)(a-\tilde{a})| + \int_0^t ||I_{\Gamma_1}(t-s)|| \cdot \left|F(u^{(j)}(s,a)) - F(u^{(j)}(s,\tilde{a}))\right| ds + \\
&\int_t^\infty ||I_{\Gamma_2}(t-s)|| \cdot \left|F(u^{(j)}(s,a)) - F(u^{(j)}(s,\tilde{a}))\right| ds \\
&\le K|a-\tilde{a}| + 2^{-m_0}\left|u^{(j)}(s,a) - u^{(j)}(s,\tilde{a})\right| \left(\int_0^t Ke^{-(\alpha+\sigma)(t-s)} ds + \int_t^\infty Ke^{\sigma(t-s)} ds\right) \\
&\le K|a-\tilde{a}| + 2^{-m_0} \cdot 3K|a-\tilde{a}| \cdot \left(\frac{K}{\alpha+\sigma} + \frac{K}{\sigma}\right) \\
&= K|a-\tilde{a}| \left(1 + 2^{-m_0}\frac{3K}{\alpha+\sigma} + 2^{-m_0}\frac{3K}{\sigma}\right) \\
&\le 3K|a-\tilde{a}| \quad (\text{Recall that } 2^{-m_0} \le \tfrac{\sigma}{4K})
\end{aligned}
$$

The proof is complete.

Acknowledgments. D. Graça was partially supported by *Fundação para a Ciência e a Tecnologia* and EU FEDER POCTI/POCI via SQIG - Instituto de Telecomunicações. N. Zhong was partially supported by the Charles Phelps Taft Memorial Fund of the University of Cincinnati.

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